

METRIC RESULTS ON THE DISCREPANCY OF SEQUENCES $(a_n\alpha)_{n\geq 1}$ MODULO ONE FOR INTEGER SEQUENCES $(a_n)_{n\geq 1}$ OF POLYNOMIAL GROWTH

CHRISTOPH AISTLEITNER AND GERHARD LARCHER

ABSTRACT. An important result of H. Weyl states that for every sequence $(a_n)_{n\geq 1}$ of distinct positive integers the sequence of fractional parts of $(a_n\alpha)_{n\geq 1}$ is uniformly distributed modulo one for almost all α . However, in general it is a very hard problem to calculate the precise order of convergence of the discrepancy of $(\{a_n\alpha\})_{n\geq 1}$ for almost all α . In particular it is very difficult to give sharp lower bounds for the speed of convergence. Until now this was only carried out for lacunary sequences $(a_n)_{n\geq 1}$ and for some special cases such as the Kronecker sequence $(\{n\alpha\})_{n\geq 1}$ or the sequence $(\{n^2\alpha\})_{n\geq 1}$. In the present paper we answer the question for a large class of sequences $(a_n)_{n\geq 1}$ including as a special case all polynomials $a_n = P(n)$ with $P \in \mathbb{Z}[x]$ of degree at least 2.

1. INTRODUCTION

In the present paper we will study distribution properties of sequences $(x_n)_{n\geq 1}$ of the form $x_n = \{a_n\alpha\}$ in the unit interval, where α is a given real and $(a_n)_{n\geq 1}$ is a given sequence of integers, and where $\{\cdot\}$ denotes the fractional part. In particular we are interested in the behavior of the discrepancy D_N of these sequences from a metric point of view.

The discrepancy D_N of a sequence $(x_n)_{n\geq 1}$ in $[0, 1)$ is given by

$$D_N = \sup_{0 \leq a < b \leq 1} \left| \frac{A_N([a, b))}{N} - (b - a) \right|$$

where $A_N([a, b)) = \#\{1 \leq n \leq N \mid x_n \in [a, b)\}$. The sequence $(x_n)_{n\geq 1}$ is uniformly distributed in $[0, 1]$ if and only if $\lim_{N \rightarrow \infty} D_N = 0$. It was shown by H. Weyl [17] that for every sequence $(a_n)_{n\geq 1}$ of distinct positive integers the sequence $(\{a_n\alpha\})_{n\geq 1}$ is uniformly distributed for almost all α . In [4] R.C. Baker gave a corresponding discrepancy estimate: Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers. Then for almost all α for the discrepancy D_N of $(\{a_n\alpha\})_{n\geq 1}$ we have

$$(1) \quad ND_N = \mathcal{O}\left(N^{\frac{1}{2}} (\log N)^{\frac{3}{2} + \varepsilon}\right)$$

The first author is supported by a Schrödinger scholarship of the Austrian Research Foundation (FWF). The second author is supported by the Austrian Science Fund (FWF): Project F5507-N26, which is part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

for all $\varepsilon > 0$. It is known that this result is best possible, up to logarithmic terms, as a generic result covering *all* strictly increasing integer sequences $(a_n)_{n \geq 1}$ (see [5]). However, in most cases of interesting particular sequences $(a_n)_{n \geq 1}$ no metric lower bounds for the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ are known at all, and in these cases it is totally unclear how close the generic upper bound in (1) comes to the “correct” metric order of the discrepancy.

One case where the precise metric asymptotic order of the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ is known is the case when $(a_n)_{n \geq 1}$ is a lacunary sequence, i.e., if $\frac{a_{n+1}}{a_n} \geq 1 + \delta$ for some constant $\delta > 0$. For this case, where certain independence properties of the sequence $(\{a_n \alpha\})_{n \geq 1}$ can be used, W. Philipp [16] proved that for almost all α

$$(2) \quad \frac{1}{4\sqrt{2}} \leq \limsup_{N \rightarrow \infty} \frac{ND_N}{\sqrt{2N \log \log N}} \leq c_\delta$$

holds. Even more precise results were obtained by K. Fukuyama [10], taking into account the number-theoretic structure of $(a_n)_{n \geq 1}$. It is interesting to note that the asymptotic result in (2) is in accordance with the Chung–Smirnov law of the iterated logarithm, which prescribes a discrepancy of order $(N \log \log N)^{-1/2}$ for the discrepancy of a “random” sequence in the unit interval, almost surely.

For the case $a_n = n$, that is for the Kronecker sequence, it follows from results of Khintchine [12] in the metric theory of continued fractions that

$$(3) \quad ND_N = \mathcal{O}(\log N (\log \log N)^{1+\varepsilon})$$

and

$$(4) \quad ND_N = \Omega(\log N \log \log N)$$

hold for every $\varepsilon > 0$ for almost all α . Of course the same result also holds in the case when a_n is a polynomial of degree 1 in n .

For the case $a_n = n^2$ it follows from a result of Fiedler, Jurkat and Körner [8] that

$$ND_N = \Omega\left(N^{\frac{1}{2}}(\log N)^{\frac{1}{4}}\right)$$

holds.

Apart from the case of lacunary $(a_n)_{n \geq 1}$, the classical case of the pure Kronecker sequence $(\{n\alpha\})_{n \geq 1}$, and from the example $(\{n^2\alpha\})_{n \geq 1}$ only for a few further examples the precise metric order of the discrepancy of $(\{a_n \alpha\})_{n \geq 1}$ is known. A very interesting special example was given recently in [3]. Here for the first time an example for $(a_n)_{n \geq 1}$ was given where the metric order of ND_N is strictly between N^ε and $N^{\frac{1}{2}-\varepsilon}$. Take for $(a_n)_{n \geq 1}$ the sequence of integers with an even sum of digits in base 2. Then for almost all α we have

$$ND_N = \mathcal{O}(N^{\kappa+\varepsilon})$$

and

$$ND_N = \Omega(N^{\kappa-\varepsilon})$$

for all $\varepsilon > 0$, where κ is a constant of the form $\kappa = 0,404\dots$. Interestingly, the precise value of the constant κ is still unknown; for details see [3] and [9].

Until now, however, to the best of our knowledge, nothing was known on the exact metric order of the discrepancy of the sequences $(\{n^k\alpha\})_{n\geq 1}$ for $k \geq 3$ or related polynomial sequences, apart from the general upper bound in (1). In particular, it seems that no non-trivial lower bound whatsoever was known in this case.

In this paper we will show that for a large class of sequences $(a_n)_{n\geq 1}$ the discrepancy D_N of $(\{a_n\alpha\})_{n\geq 1}$ has an asymptotic order of roughly $\frac{1}{\sqrt{N}}$. This class in particular contains all sequences $a_n = P(n)$ with $P(n) \in \mathbb{Z}[x]$ of degree larger or equal 2. Consequently, together with the results from (3) and (4), we have now a fairly complete understanding of the metric discrepancy behavior of sequences generated by polynomials with integer coefficients.

Our main result is the following:

Theorem 1. *Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $d \geq 2$ and let $(m_n)_{n\geq 1}$ be an arbitrary sequence of pairwise different integers with $|m_n| \leq n^t$ for some $t \in \mathbb{N}$ and all $n \geq n(t)$. Then for the discrepancy D_N of the sequence $(\{P(m_n)\alpha\})_{n\geq 1}$ we have for almost all α*

$$ND_N \geq N^{\frac{1}{2}-\varepsilon}$$

for all $\varepsilon > 0$ and for infinitely many N .

As a direct consequence of Theorem 1 (by choosing $m_n = n$) we obtain the following corollary.

Corollary 1. *Let $P \in \mathbb{Z}[x]$ be a polynomial of degree $d \geq 2$. Then for the discrepancy D_N of the sequence $(\{P(n)\alpha\})_{n\geq 1}$ we have for almost all α*

$$ND_N \geq N^{\frac{1}{2}-\varepsilon}$$

for all $\varepsilon > 0$ and for infinitely many N .

The same estimate for example also holds if we choose $m_n = p_n$, the n -th prime, or $m_n = [n\beta]$ for some fixed $\beta > 1$.

Theorem 1 is a consequence of the more general

Theorem 2. *Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ be a function with $|f(n)| \leq n^t$ for some $t \in \mathbb{N}$ and all $n \geq n(t)$. Set*

$$A_f(n) := \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid f(x) + f(y) = n\}$$

and

$$\tilde{A}_f(n) := \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid f(x) - f(y) = n\}$$

and assume that we have $A_f(n) = \mathcal{O}(|n|^\varepsilon)$ as $|n| \rightarrow \infty$ or $\tilde{A}_f(n) = \mathcal{O}(|n|^\varepsilon)$ as $n \rightarrow \infty$ for all $\varepsilon > 0$. Then for almost all α for the discrepancy of the sequence $(\{f(n)\alpha\})_{n \geq 1}$ we have

$$ND_N \geq N^{\frac{1}{2}-\varepsilon}$$

for all $\varepsilon > 0$ and for infinitely many N .

Theorem 2 is a consequence of the following result which implicitly already was used in [3], and which shows that under certain conditions on $(a_n)_{n \geq 1}$ the metric behavior of the discrepancy of $(\{a_n\alpha\})_{n \geq 1}$ is determined by the L1-norm of the exponential sums in $a_n\alpha$.

Theorem 3. *Let $(a_n)_{n \geq 1}$ be a sequence of integers such that for some $t \in \mathbb{N}$ we have $|a_n| \leq n^t$ for all n large enough. Assume there exist a number $\tau \in (0, 1)$ and a strictly increasing sequence $(B_L)_{L \geq 1}$ of positive integers with $(B')^L \leq B_L \leq B^L$ for some reals B', B with $1 < B' < B$, such that for all $\varepsilon > 0$ and all $L > L(\varepsilon)$ we have*

$$(5) \quad \int_0^1 \left| \sum_{n=1}^{B_L} e^{2\pi i a_n \alpha} \right| d\alpha > B_L^{\tau-\varepsilon}.$$

Then for almost all $\alpha \in [0, 1)$ for all $\varepsilon > 0$ for the discrepancy D_N of the sequence $(\{a_n\alpha\})_{n \geq 1}$ we have

$$ND_N > N^{\tau-\varepsilon}$$

for infinitely many N .

For the proof of Theorem 3 we need an auxiliary result from metric Diophantine approximation which is of some interest on its own. We state this result as Theorem 4. In the statement of the theorem and in the sequel, \mathbb{P} denotes the one-dimensional Lebesgue measure.

Theorem 4. *Let $(R_L)_{L \geq 0}$ be a sequence of measurable subsets of $[0, 1)$, with $\mathbb{P}(R_L) \geq \frac{1}{B^L}$ for some constant $B \in \mathbb{R}^+$ and such that each R_L is the disjoint union of at most A^L intervals for some $A \in \mathbb{R}^+$. Then for almost all $\alpha \in [0, 1)$ for every $\eta > 0$ there are infinitely many integers h_L with*

$$h_L \leq (1 + \eta)^L \frac{1}{\mathbb{P}(R_L)}$$

and

$$\{h_L\alpha\} \in R_L.$$

Theorem 1 seduces to the hypothesis that a polynomial growth behavior of $(a_n)_{n \geq 1}$ of degree at least 2 always implies a metric lower bound of order $N^{\frac{1}{2}-\varepsilon}$ for the discrepancy of $(\{a_n\alpha\})_{n \geq 1}$. Here by polynomial growth behavior of degree d we mean not only that $a_n \geq cn^d$ for some $c > 0$ and all n large enough, but that the stronger local condition

$$\frac{a_{n+1}}{a_n} > 1 + \frac{c}{a_n^{\frac{1}{d}}}$$

holds for some $c > 0$ and all n large enough.¹ However, this hypothesis does not hold as is shown in the following Theorem 5.

Theorem 5. *For every integer $d \geq 1$ there is a strictly increasing sequence $(a_n)_{n\geq 1}$ of integers with*

$$\frac{a_{n+1}}{a_n} > 1 + \frac{c}{a_n^{\frac{1}{d}}}$$

for some $c > 0$ and all $n \in \mathbb{N}$ such that for almost all α the discrepancy D_N of the sequence $(\{a_n\alpha\})_{n\geq 1}$ satisfies

$$ND_N = \mathcal{O}((\log N)^{2+\varepsilon})$$

for all $\varepsilon > 0$.

We conclude this section with some remarks on our theorems and some open problems. As noted before, by the Chung–Smirnov law of the iterated logarithm for a “random” sequence in the unit interval the quantity ND_N typically is of order $(N \log \log N)^{1/2}$; thus, by (1) and by Theorems 1 and 2, roughly speaking the sequences investigated in the present paper exhibit nearly random behavior. Our method only allows us to obtain discrepancy results with an error of order N^ε ; it would be very interesting to improve these error estimates to logarithmic terms, such as those in (1).

The fact that until now lower bounds have been almost non-existent in metric discrepancy theory comes from the fact that such lower bounds cannot be directly deduced from the second Borel–Cantelli lemma, which requires *independence*. In contrast in the first Borel–Cantelli lemma, which is used to prove upper bounds, only estimates for the size of exceptional sets are required, which can be deduced from simple moment estimates and Markov’s inequality. The lower bounds for the discrepancy of lacunary sequences come from an approximation of $(\{a_n\alpha\})_{n\geq 1}$ by an *independent* random system, which is possible because of the fast growth of $(a_n)_{n\geq 1}$. This is not possible for slowly growing $(a_n)_{n\geq 1}$ as in the case of the present paper. Instead we have developed a method which uses estimates for the L1-norm of exponential sum plus an *quasi-independence* property of the dilated functions $(\{ha_n\alpha\})$ for $h = 1, 2, \dots$. In the present paper the required lower bounds for L1-norms are deduced from upper bounds for L4-norms, which can be obtained by simply counting the number of solutions of certain linear equations (see Section 3). The quasi-independence property is established using a transition from norm-estimates for sums of dilated functions to certain sums involving greatest common divisors (GCD sums), together with recent strong bounds for such GCD sums (see Section 2). We believe that the relevance of this method goes far beyond the applications in the present paper, as the first general method for proving lower bounds in metric discrepancy theory beyond the well-known and totally different methods for lacunary sequences. We also want to point

¹Note that for $a_n = n^d$ we have

$$1 + \frac{d}{a_n^{\frac{1}{d}}} < \frac{a_{n+1}}{a_n} < 1 + \frac{d+1}{a_n^{\frac{1}{d}}}$$

for all n large enough.

out that these topics are related to a problem in the metric theory of Diophantine approximation posed by LeVeque [15] which is still unsolved; see the last section of [3] for details.

The remaining part of this paper is organized as follows. In Section 2 we first prove Theorem 4 and then Theorem 3. In Section 3 we deduce Theorem 2 and Theorem 1 from Theorem 3. Finally in Section 4 we prove Theorem 5.

2. PROOFS OF THEOREM 4 AND THEOREM 3

Proof of Theorem 4. Let $1_L(\alpha)$ denote the indicator function of the set R_L , extended with period 1, and $\mathbb{I}_L(\alpha) := 1_L(\alpha) - \int_0^1 1_L(\omega) d\omega$.

We have

$$\int_0^1 1_L(\alpha) d\alpha = \mathbb{P}(R_L)$$

and

$$\int_0^1 \mathbb{I}_L(\alpha) d\alpha = 0,$$

and, by assumption, for the total variation $\text{Var } \mathbb{I}_L$ of \mathbb{I}_L on $[0, 1)$ we have

$$(6) \quad \text{Var } \mathbb{I}_L \leq 2A^L.$$

Furthermore we have

$$(7) \quad \|\mathbb{I}_L\|_2^2 := \int_0^1 (\mathbb{I}_L(\alpha))^2 d\alpha = \mathbb{P}(R_L)(1 - \mathbb{P}(R_L)) \leq \mathbb{P}(R_L).$$

Let $H_L := \left\lfloor (1 + \eta)^L \frac{1}{\mathbb{P}(R_L)} \right\rfloor$. Then by definition

$$(8) \quad \sum_{h \leq H_L} 1_L(h\alpha) = \sum_{h \leq H_L} \int_0^1 1_L(\omega) d\omega + \sum_{h \leq H_L} \mathbb{I}_L(h\alpha),$$

and

$$(9) \quad \sum_{h \leq H_L} \int_0^1 1_L(\omega) d\omega = H_L \mathbb{P}(R_L) \geq \frac{1}{2} (1 + \eta)^L$$

for L large enough.

We write

$$\mathbb{I}_L(\alpha) \sim \sum_{j=1}^{\infty} (u_j \cos 2\pi j\alpha + v_j \sin 2\pi j\alpha)$$

for the Fourier series of \mathbb{I}_L . From (6) and a classical inequality for the size of the Fourier coefficients of functions of bounded variation (see for example [18], p. 48) we have

$$(10) \quad |u_j| \leq \frac{\text{Var } \mathbb{I}_L}{2j} \leq \frac{A^L}{j} \quad \text{and}$$

$$|v_j| \leq \frac{A^L}{j}.$$

We split the function \mathbb{I}_L into an even and an odd part (that is, into a cosine- and a sine-series). In the sequel, we consider only the even part; the odd part can be treated in exactly the same way. Set $G_L := (AB(1+\eta))^{2L}$, where A and B are the constants from the statement of the theorem. Let $p_L(\alpha)$ denote the G_L -th partial sum of the Fourier series of the even part of \mathbb{I}_L , and let $r_L(\alpha)$ denote the remainder term (for simplicity of writing we assume that G_L is an integer). Then by Minkowski's inequality we have

$$(11) \quad \left\| \sum_{h=1}^{H_L} \mathbb{I}_L^{(\text{even})}(h \cdot) \right\|_2 \leq \left\| \sum_{h=1}^{H_L} p_L(h \cdot) \right\|_2 + \left\| \sum_{h=1}^{H_L} r_L(h \cdot) \right\|_2.$$

Furthermore, (10), Minkowski's inequality, and Parseval's identity imply that

$$\begin{aligned} \left\| \sum_{h=1}^{H_L} r_L(h \cdot) \right\|_2 &\leq H_L \|r_L\|_2 \\ &\leq H_L \sqrt{\sum_{j=G_L+1}^{\infty} \frac{A^{2L}}{j^2}} \\ &\leq \frac{H_L A^L}{G_L^{1/2}} \\ &\leq \frac{(1+\eta)^L A^L}{\mathbb{P}(R_L) G_L^{1/2}} \\ (12) \quad &\leq 1. \end{aligned}$$

To estimate the first term on the right-hand side of (11), we expand p_L into a Fourier series and use the orthogonality of the trigonometric system. Then we obtain

$$\begin{aligned} \left\| \sum_{h=1}^{H_L} p_L(h \cdot) \right\|_2^2 &= \sum_{\substack{k_1, k_2=1 \\ j_1 h_1 = j_2 h_2}}^{H_L} \sum_{j_1, j_2=1}^{G_L} \frac{u_{j_1} u_{j_2}}{2} \\ (13) \quad &= \sum_{j_1, j_2=1}^{G_L} \frac{u_{j_1} u_{j_2}}{2} \# \left\{ (h_1, h_2) : 1 \leq h_1, h_2 \leq H_L, j_1 h_1 = j_2 h_2 \right\}. \end{aligned}$$

To estimate the size of the sum on the right-hand side of (13), we assume that j_1 and j_2 are fixed. It turns out that we have $j_1 h_1 = j_2 h_2$ whenever

$$h_1 = w \frac{j_2}{\gcd(j_1, j_2)}, \quad h_2 = w \frac{j_1}{\gcd(j_1, j_2)} \quad \text{for some positive integer } w$$

(see Section 5 of [3] for a more detailed deduction). As a consequence we have

$$\begin{aligned}
& \#\left\{(h_1, h_2) : 1 \leq h_1, h_2 \leq H_L, j_1 h_1 = j_2 h_2\right\} \\
&= \#\left\{w \geq 1 : w \leq \min\left(\frac{H_L \gcd(j_1, j_2)}{j_2}, \frac{H_L \gcd(j_1, j_2)}{j_1}\right)\right\} \\
&= \left\lfloor \frac{H_L \gcd(j_1, j_2)}{\max(j_1, j_2)} \right\rfloor \\
&\leq \frac{H_L \gcd(j_1, j_2)}{\sqrt{j_1 j_2}}.
\end{aligned}$$

Combining this estimate with (13) we obtain

$$(14) \quad \left\| \sum_{h=1}^{H_L} p_L(h \cdot) \right\|_2^2 \leq H_L \sum_{j_1, j_2=1}^{G_L} \frac{|u_{j_1} u_{j_2}| \gcd(j_1, j_2)}{2 \sqrt{j_1 j_2}}.$$

The sum on the right-hand side of the last equation is called a *GCD sum*. It is well-known that such sums play an important role in the metric theory of Diophantine approximation; the particular sum in (14) probably appeared for the first time in LeVeque's paper [15] (see also [1] and [7]). A precise upper bound for these sums has been obtained by Hilberdink [11].² Hilberdink's result implies that there exists an absolute constant c_{abs} such that

$$\sum_{j_1, j_2=1}^{G_L} \frac{|u_{j_1} u_{j_2}| \gcd(j_1, j_2)}{2 \sqrt{j_1 j_2}} \ll \exp\left(\frac{c_{\text{abs}} \sqrt{\log(G_L)}}{\sqrt{\log \log G_L}}\right) \sum_{j=1}^{G_L} u_j^2.$$

Combining this estimate with (7) and (14) (and using Parseval's identity) we have

$$\begin{aligned}
\left\| \sum_{h=1}^{H_L} p_L(h \cdot) \right\|_2^2 &\ll H_L \exp\left(\frac{c_{\text{abs}} \sqrt{\log(G_L)}}{\sqrt{\log \log G_L}}\right) \mathbb{P}(R_L) \\
&\ll (1 + \eta)^L \exp\left(\frac{c_{\text{abs}} \sqrt{\log(G_L)}}{\sqrt{\log \log G_L}}\right),
\end{aligned}$$

and, together with (11) and (12), and with a similar argument for the odd part of \mathbb{I}_L , we obtain

$$(15) \quad \left\| \sum_{h=1}^{H_L} \mathbb{I}_L(h \cdot) \right\|_2^2 \ll (1 + \eta)^L \exp\left(\frac{c_{\text{abs}} \sqrt{\log(G_L)}}{\sqrt{\log \log G_L}}\right).$$

²The upper bounds for the GCD sums in [11] are formulated in terms of the largest eigenvalues of certain GCD matrices; since these matrices are symmetric and positive definite, the largest eigenvalue also gives an upper bound for the GCD sum. This relation is explained in detail in [2].

By Chebyshev's inequality we have

$$\mathbb{P} \left(\alpha \in [0, 1) : \left| \sum_{h=1}^{H_L} \mathbb{I}_L(h\alpha) \right| > (\log H_L) \left\| \sum_{h=1}^{H_L} \mathbb{I}_L(h\cdot) \right\|_2 \right) \leq \frac{1}{(\log H_L)^2},$$

and since $(H_L)_{L \geq 1}$ grows exponentially in L these probabilities give a convergent series when summing over L . Thus by the Borel–Cantelli lemma for almost all α only finitely many events

$$\left| \sum_{h=1}^{H_L} \mathbb{I}_L(h\alpha) \right| > (\log H_L) \left\| \sum_{h=1}^M \mathbb{I}_L(h\cdot) \right\|_2$$

happen, which by (15) implies that

$$\left| \sum_{h=1}^{H_L} \mathbb{I}_L(h\alpha) \right| \ll (1 + \eta)^{L/2} \exp \left(\frac{\hat{c}_{\text{abs}} \sqrt{L}}{\sqrt{\log L}} \right)$$

for some absolute constant \hat{c}_{abs} . Comparing this upper bound with (8) and using (9) we conclude that

$$\sum_{h=1}^{H_L} \mathbf{1}_L(h\alpha) \gg (1 + \eta)^L \quad \text{as } L \rightarrow \infty$$

for almost all α . In particular we have

$$\sum_{L=1}^{\infty} \sum_{h=1}^{H_L} \mathbf{1}_L(h\alpha) = \infty$$

for almost all α , and the result follows. \square

Proof of Theorem 3. By the Koksma–Hlawka inequality (see for example [6] or [13]) for every positive integer H we have

$$(16) \quad ND_N \geq \frac{1}{4H} \left| \sum_{n=1}^N e^{2\pi i H a_n \alpha} \right|.$$

Let

$$f_L(\alpha) := \left| \sum_{n=1}^{B_L} e^{2\pi i a_n \alpha} \right|.$$

We will show that for any given $\varepsilon > 0$ for almost all α there are infinitely many L such that there exists a positive integer h_L with

$$(17) \quad \frac{1}{h_L} f_L(h_L \alpha) \gg B_L^{\tau - \varepsilon},$$

which together with (16) implies the conclusion of the theorem.

Let $\varepsilon > 0$ with $\varepsilon < \frac{\tau}{2}$ be given. From the definition of $f_L(\alpha)$ and the fact that $|a_n| \leq n^t$ for large n it is easily seen that for sufficiently large L we have

$$\begin{aligned} |f_L(\alpha_1) - f_L(\alpha_2)| &\leq 2\pi B_L B_L^t |\alpha_1 - \alpha_2| \\ (18) \qquad \qquad \qquad &\leq 2\pi (B^{1+t})^L |\alpha_1 - \alpha_2|. \end{aligned}$$

Now let $g_L(\alpha)$ be the function defined by

$$\begin{aligned} g_L(\alpha) &:= f_L\left(j(B^{1+t})^{-L}\right) \quad \text{for } \alpha \in \left[j(B^{1+t})^{-L}, (j+1)(B^{1+t})^{-L}\right) \\ &\quad \text{and for } j = 0, 1, \dots, (B^{1+t})^L - 1 \end{aligned}$$

(for simplicity of writing we assume that $(B^{1+t})^L$ is an integer). Then by (18) we have

$$|g_L - f_L| \leq 2\pi,$$

which means that it is sufficient to prove (17) with f_L replaced by g_L . By construction the function g_L can be written as a sum of at most $(B^{1+t})^L$ indicator functions of disjoint intervals.

Let

$$Q := \left\lfloor \frac{1 - \tau + 2\varepsilon}{3\varepsilon} \right\rfloor + 1$$

and

$$\Delta_i := B_L^{\tau-2\varepsilon} B_L^{3\varepsilon i} \quad \text{for } i = 0, 1, \dots, Q.$$

Note that $\Delta_0 = B_L^{\tau-2\varepsilon}$ and $B_L \leq \Delta_Q \leq B_L^{1+3\varepsilon}$. Define

$$M_L^i := \left\{ \alpha \in [0, 1) \text{ with } \Delta_i < |g_L(\alpha)| \leq \Delta_{i+1} \right\}$$

for $i = 0, \dots, Q-1$. Then by (5) we have

$$\begin{aligned} \sum_{i=0}^{Q-1} \mathbb{P}\left(M_L^{(i)}\right) \Delta_{i+1} + \left(1 - \sum_{i=0}^{Q-1} \mathbb{P}\left(M_L^{(i)}\right)\right) \Delta_0 &\geq \int_0^1 |g_L(\alpha)| d\alpha \\ &> \frac{B_L^{\tau-\varepsilon}}{2} \end{aligned}$$

for sufficiently large L .

By $\Delta_0 = B_L^{\tau-2\varepsilon}$ we have

$$\sum_{i=0}^{Q-1} \mathbb{P}\left(M_L^{(i)}\right) \Delta_{i+1} \geq \frac{B_L^{\tau-\varepsilon}}{4}$$

for L large enough. Consequently for every L large enough there is an $i_L \in \{0, \dots, Q-1\}$ with

$$\Delta_{i_L+1} \mathbb{P}\left(M_L^{(i_L)}\right) \geq \frac{B_L^{\tau-\varepsilon}}{4Q},$$

which implies that

$$\mathbb{P} \left(M_L^{(i_L)} \right) \geq \frac{1}{4Q} \frac{B_L^{\tau-\varepsilon}}{\Delta_{i_L+1}} \geq \frac{1}{4Q} \frac{1}{B_L^{1-\tau+4\varepsilon}} \geq \left(\frac{1}{B^{1-\tau+5\varepsilon}} \right)^L$$

for L large enough. Note that, as a consequence of the construction of g_L , the set $M_L^{(i_L)}$ is always a union of at most $(B^{1+t})^L$ intervals. By Theorem 4 we conclude that for almost all α for all $\eta > 0$ there are infinitely many integers h_L with $h_L < (1+\eta)^L \frac{1}{\mathbb{P}(M_L^{(i_L)})}$ and

$\{h_L \alpha\} \in M_L^{(i_L)}$. Consequently for almost all α we have

$$\begin{aligned} \frac{1}{h_L} |g_L(h_L \alpha)| &\geq \frac{1}{(1+\eta)^L} \mathbb{P} \left(M_L^{(i_L)} \right) \Delta_{i_L} \\ &\geq \frac{1}{(1+\eta)^L} \frac{\Delta_{i_L}}{\Delta_{i_L+1}} \mathbb{P} \left(M_L^{(i_L)} \right) \\ &\geq \frac{1}{(1+\eta)^L} \frac{1}{B_L^{3\varepsilon}} \frac{B_L^{\tau-\varepsilon}}{4Q} \\ &\geq B_L^{\tau-5\varepsilon} \end{aligned}$$

for η small enough and for infinitely many L , since by assumption $B_L \geq (B')^L$ for some constant $B' > 1$. This proves the theorem. \square

3. PROOFS OF THEOREM 2 AND THEOREM 1

Proof of Theorem 2. By Theorem 3 it suffices to show that

$$\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right| d\alpha > N^{\frac{1}{2}-\varepsilon}$$

for all $\varepsilon > 0$ and N large enough.

Using a standard trick, by Hölder's inequality with

$$F(\alpha) := \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^{\frac{2}{3}} \quad \text{and} \quad G(\alpha) := \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^{\frac{4}{3}}$$

and with $p = \frac{3}{2}$ and $q = 3$ we have

$$\begin{aligned} \int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^2 d\alpha &= \int_0^1 F(\alpha) G(\alpha) d\alpha \\ &\leq \left(\int_0^1 (F(\alpha))^{\frac{3}{2}} d\alpha \right)^{\frac{2}{3}} \left(\int_0^1 (G(\alpha))^3 d\alpha \right)^{\frac{1}{3}} \end{aligned}$$

$$= \left(\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right| d\alpha \right)^{\frac{2}{3}} \left(\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^4 d\alpha \right)^{\frac{1}{3}},$$

hence

$$(19) \quad \int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right| d\alpha \geq \frac{\left(\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^2 d\alpha \right)^{\frac{3}{2}}}{\left(\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^4 d\alpha \right)^{\frac{1}{2}}}.$$

Now

$$\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right|^2 d\alpha = \sum_{\substack{1 \leq m, n \leq N, \\ f(m) = f(n)}} 1 \geq N,$$

and

$$\begin{aligned} \int_0^1 \left| \sum_{h=1}^N e^{2\pi i f(h)\alpha} \right|^4 d\alpha &= \sum_{\substack{1 \leq k, l, m, n \leq N, \\ f(k) + f(l) = f(m) + f(n)}} 1 \\ &= \sum_a A_f^2(a) = \sum_a \tilde{A}_f^2(a), \end{aligned}$$

where summation in the penultimate sum is over all integers $a \in [-2N^t, 2N^t]$ such that there exist k, l with $1 \leq k, l \leq N$ and $f(k) + f(l) = a$, and summation in the last sum is over all $a \in [-2N^t, 2N^t]$ such that there exist k, l with $1 \leq k, l \leq N$ and $f(k) - f(m) = a$. By assumption we either have

$$\begin{aligned} \sum_a A_f^2(a) &\leq \left(\max_{a \in [-2N^t, 2N^t]} \{A_f^2(a)\} \right) \# \{a : \exists 1 \leq k, l \leq N \text{ with } f(k) + f(l) = a\} \\ &\leq (2N^t)^{2\frac{\varepsilon}{2t}} N^2 \\ &\leq c(\varepsilon) N^{2+\varepsilon} \end{aligned}$$

for all $\varepsilon > 0$ and all $N \geq N(\varepsilon)$, or we have the corresponding estimate for $\sum_a \tilde{A}_f^2(a)$.

Hence by (19) for all $\varepsilon > 0$ and for $N \geq N(\varepsilon)$ we have

$$\int_0^1 \left| \sum_{n=1}^N e^{2\pi i f(n)\alpha} \right| d\alpha \geq \frac{N^{\frac{3}{2}}}{(c(\varepsilon) N^{2+\varepsilon})^{\frac{1}{2}}} = \frac{1}{c(\varepsilon)^{\frac{1}{2}}} N^{\frac{1}{2}-\frac{\varepsilon}{2}}$$

and the result follows. \square

Proof of Theorem 1. By Theorem 2 it suffices to show that $f(n) := P(m_n)$ satisfies

$$\tilde{A}_f(n) = \mathcal{O}(|n|^\varepsilon)$$

for all $\varepsilon > 0$.

Let first $m_n = n$. By assumption f is of degree $d \geq 2$, hence there exists a non-constant polynomial $q \in \mathbb{Z}[x, y]$ such that $f(x) - f(y) = (x - y)q(x, y)$. So, if $f(x) - f(y) = n$ for some non-zero integer n , it follows that $x - y$ is a divisor t of n , hence $q(x, x - t) = \frac{n}{t}$. This last equation has at most $d - 1$ solutions x . Since n has $\mathcal{O}(|n|^\varepsilon)$ divisors t for all $\varepsilon > 0$, the assertion $\tilde{A}_f(n) = \mathcal{O}(|n|^\varepsilon)$ follows.

For arbitrary (m_n) the result follows trivially from the special result for $m_n = n$. \square

4. PROOF OF THEOREM 5

For the proof of Theorem 5 we will make use of the following Lemma on continued fractions.

Lemma 1. *For $x \in \mathbb{R}$ let $b_m(x)$ denote the m -th continued fraction coefficient of x . For any integer $\beta \geq 2$ and $L \in \mathbb{N}$ let*

$$S_L(\alpha) := \sum_{k=1}^L \sum_{m=1}^L b_m(\beta^k \alpha).$$

Then for almost all α we have

$$S_L(\alpha) = \mathcal{O}(L^{2+\varepsilon})$$

for every $\varepsilon > 0$.

Proof. This is a special case of Lemma 2 in [14]. \square

Proof of Theorem 5. For given integer $d \geq 1$ we now construct a sequence $(b_n)_{n \geq 1}$ which satisfies the properties stated in Theorem 5.

Let $(b_n)_{n \geq 1}$ be the strictly increasing sequence of integers running through the integers

$$2^{dk} + j2^{dk+d-k}$$

for $j = 0, \dots, 2^k - 1$ and $k = 1, 2, 3, \dots$

We have

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{2^{dk} + j2^{dk+d-k} + 2^{dk+d-k}}{2^{dk} + j2^{dk+d-k}} \\ &= 1 + \frac{1}{2^{k-d} + j} \\ &\geq 1 + \frac{1}{(2^{dk} + j2^{dk+d-k})^{\frac{1}{d}}} \\ &= 1 + \frac{1}{b_n^{\frac{1}{d}}}, \end{aligned}$$

since (as is easily checked)

$$2^{k-d} + j \leq (2^{dk} + j2^{dk+d-k})^{\frac{1}{d}}$$

for $j = 0$ and $j = 2^k$ and hence for all $j = 0, \dots, 2^k$. So $(b_n)_{n \geq 1}$ has polynomial growth behavior of degree at least d .

Let N be given and k_0 and j_0 such that $b_N = 2^{dk_0} + j_0 2^{dk_0+d-k_0}$ for some $j_0 \in \{0, \dots, 2^{k_0} - 1\}$. By standard techniques from the theory of uniform distribution theory for the discrepancy D_N of $(\{b_n \alpha\})_{n \geq 1}$ we have

$$ND_N \leq \sum_{k=1}^{k_0} 2^k D^{(k)},$$

where by $D^{(k)}$ we denote the discrepancy of the subsequence

$$(\{2^{dk} \alpha + j 2^{(d-1)k} 2^d \alpha\}); \quad j = 0, 1, \dots, 2^k - 1.$$

For this sequence it is also well known (see for example [13]) that

$$2^k D^{(k)} \leq c_{\text{abs}} \sum_{m=1}^{2^k} b_m \left((2^{(d-1)})^k 2^d \alpha \right),$$

with an absolute constant c_{abs} , hence

$$ND_N \leq c_{\text{abs}} \sum_{k=1}^{2k_0} \sum_{m=1}^{2k_0} b_m (\beta^k 2^d \alpha),$$

where $\beta = 2^{(d-1)}$ and $b_m(x)$ again denotes the m -th continued fraction coefficient of x . By Lemma 1 therefore for almost all $2^d \alpha$ (and consequently also for almost all α) we have

$$ND_N = \mathcal{O}(k_0^{2+\varepsilon}) = \mathcal{O}((\log N)^{2+\varepsilon}) \quad \text{as } N \rightarrow \infty$$

for all $\varepsilon > 0$ (note that $N \geq 2^{k_0}$). This proves the theorem. \square

REFERENCES

- [1] C. Aistleitner, I. Berkes, and K. Seip. GCD sums from Poisson integrals and systems of dilated functions. *J. Eur. Math. Soc.*, 17(6):1517–1546, 2015.
- [2] C. Aistleitner, I. Berkes, K. Seip, and M. Weber. Convergence of series of dilated functions and spectral norms of GCD matrices. *Acta Arith.*, 168(3):221–246, 2015.
- [3] C. Aistleitner, R. Hofer and G. Larcher. On parametric Thue-Morse Sequences and Lacunary Trigonometric Products. Submitted, 2015. Available at <http://arxiv.org/abs/1502.06738>.
- [4] R. C. Baker. Metric number theory and the large sieve. *J. London Math. Soc. (2)*, 24(1):34–40, 1981.
- [5] I. Berkes and W. Philipp. *J. London Math. Soc. (2)*, 50(3):454–464, 1994.
- [6] M. Drmota and R. F. Tichy. *Sequences, discrepancies and applications*, volume 1651 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.
- [7] T. Dyer and G. Harman. Sums involving common divisors. *J. London Math. Soc. (2)*, 34(1):1–11, 1986.
- [8] H. Fiedler, W. Jurkat, and O. Körner. Asymptotic expansions of finite theta series. *Acta Arith.*, 32:129–146, 1977.

- [9] E. Fouvry and C. Mauduit. Sommes des chiffres et nombres presque premiers. *Math. Ann.*, 305(3):571–599, 1996.
- [10] K. Fukuyama. The law of the iterated logarithm for discrepancies of $\{\theta^n x\}$. *Acta Math. Hungar.*, 118(1-2):155–170, 2008.
- [11] T. Hilberdink. An arithmetical mapping and applications to Ω -results for the Riemann zeta function. *Acta Arith.*, 139(4):341–367, 2009.
- [12] A. Khintchine. Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen. *Math. Ann.*, 92:115–125, 1924.
- [13] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
- [14] G. Larcher. Probabilistic Diophantine approximation and the distribution of Halton–Kronecker sequences. *J. Complexity*, 29:397–423, 2013.
- [15] W. J. LeVeque. On the frequency of small fractional parts in certain real sequences III. *J. Reine Angew. Math.*, 202:215–220, 1959.
- [16] W. Philipp. Limit theorems for lacunary series and uniform distribution mod 1. *Acta Arith.*, 26(3):241–251, 1974/75.
- [17] H. Weyl. Über die Gleichverteilung von Zahlen modulo Eins. *Math. Ann.*, 77:313–352, 1916.
- [18] A. Zygmund. *Trigonometric series. Vol. I, II*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1979 edition.

INSTITUTE OF FINANCIAL MATHEMATICS AND APPLIED NUMBER THEORY, UNIVERSITY LINZ
E-mail address: `aistleitner@math.tugraz.at`

INSTITUTE OF FINANCIAL MATHEMATICS AND APPLIED NUMBER THEORY, UNIVERSITY LINZ
E-mail address: `gerhard.larcher@jku.at`